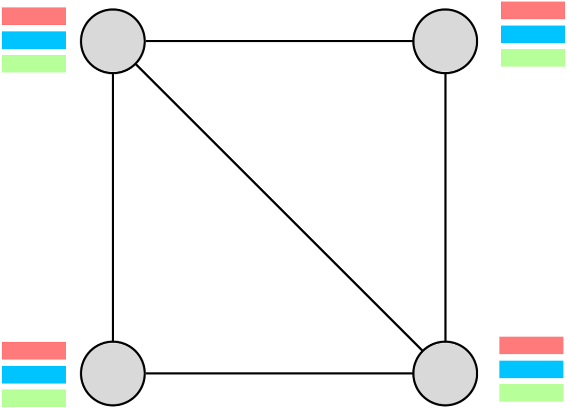
Flexible list coloring and maximum average degree

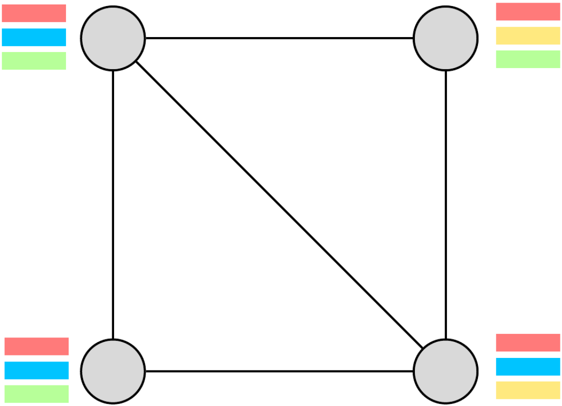
Peter Bradshaw
(contains joint work with Richard Bi)

Sun Yat Sen University

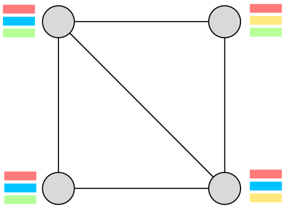
Proper coloring



List coloring

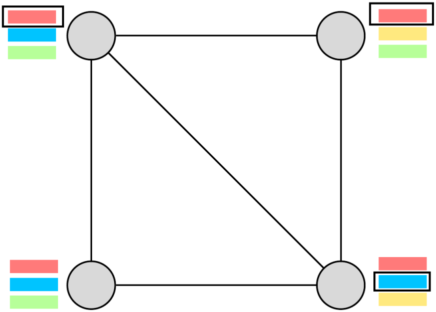


Choosability



If G has a list coloring for every assignment of lists of size k, then G is *k-choosable*.

List coloring with preferences

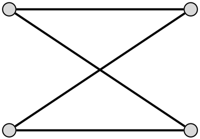


We say that G is ϵ-*flexibly* k-*choosable* if for every assignment of lists of size k, and for every set of coloring preferences, there exists a list coloring of G satisfying an ϵ proportion of all preferences.

Examples

- An independent set is 1-flexibly 1-choosable.
- A path is $\frac{1}{2}$-flexibly 2-choosable.
- The square of a path is $\frac{1}{3}$-flexibly 3-choosable.

A non-example



$K_{2,2}$ is 2-choosable but is not ϵ-flexibly 2-choosable for any $\epsilon > 0$.

Some positive results

Theorem (PB, Masařík, Stacho)

If G has maximum degree $\Delta \geq 3$ and no $K_{\Delta+1}$ subgraph, then G is $\frac{1}{6\Delta}$-flexibly Δ-choosable.

Some positive results

Theorem (PB, Masařík, Stacho)

If G has treewidth 2, then G is $\frac{1}{3}$-flexibly 3-choosable.

Some positive results

Theorem (Dvořák, Masařík, Musílek, Pangrác)

There exists a value $\epsilon > 0$ such that if G is planar and triangle-free, then G is ϵ-flexibly 4-choosable.

How to prove ϵ-flexible k-choosability

Try to show that each k-assignment L admits a good probability distribution on L colorings:

- Suppose that for each $v \in V(G)$, each $c \in L(v)$ is used at v with probability at least ϵ.
- Then, G is ϵ-flexibly k-choosable.

Meta question

Question (Dvořák, Norin, Postle 2019)

Suppose $\mathcal{G}$ is a graph class consisting of k-choosable graphs. Does there exist a value $\epsilon > 0$ such that every graph in $\mathcal{G}$ is ϵ-flexibly k-choosable?

Degenerate graphs

A graph G is *d-degenerate* if every nonempty subgraph of G has a vertex of degree at most d.

- Trees are 1-degenerate.

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- Triangle-free planar graphs are 3-degenerate.

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A d-degenerate graph is $(d+1)$-choosable.

A more specific question

Question (Dvořák, Norin, Postle 2019)

Suppose $\mathcal{G}$ is the class of d-degenerate graphs. Does there exist a value $\epsilon > 0$ such that every graph in $\mathcal{G}$ is ϵ-flexibly $(d+1)$-choosable?

The *average degree* of a graph G, written $ad(G)$, is the mean taken over all values $\deg(v)$ for $v \in V(G)$.

The *maximum average degree* of G is the maximum value $ad(H)$ taken over all nonempty subgraphs H of G.

If G has maximum average degree $< d + 1$, then G is d-degenerate.

An even more specific question

Question

Suppose $\mathcal{G}$ *is the class of graphs with maximum average degree* $< d + 1$*. Does there exist a value* $\epsilon > 0$ *such that every graph in* $\mathcal{G}$ *is* ϵ*-flexibly* $(d + 1)$*-choosable?*

An even more specific answer

Theorem (Bi, PB)

If G has maximum average degree less than 3, then G is 2^{-30}-flexibly 3-choosable.

This result improves:

Theorem (Dvořák, Masařík, Musílek, Pangrác)

If G is planar with girth at least 6, then G is ϵ-flexibly 3-choosable.

Reducible subgraphs

In graph coloring, a *reducible subgraph* is a subgraph that can be colored last.

Let G be a graph, and let H be an induced subgraph. We say that H is *reducible* if:

- Each $v \in V(H)$ has at most one neighbor in $G \setminus H$;
- For every assignment of lists of size 3 on H and coloring of $G \setminus H$, there exists a distribution on colorings of H, so that each available color is used at its vertex with probability at least $\alpha > 0$.

Lemma (Dvořák, Masařík, Musílek, Pangrác)

There exists a value $\epsilon > 0$ *such that if every induced subgraph of* G *has a reducible subgraph, then* G *is* ϵ*-flexibly 3-choosable.*

A simple application of the framework

Proposition (Dvořák, Norin, and Postle)

If G has maximum average degree < 2.4, then G is ϵ-flexibly 3-choosable.

Suppose G has no reducible subgraph.

- Let each v receive charge $\deg(v) - 2.4$.
- Each deg 2 vertex takes charge 0.2 from each neighbor.
- The final charge is nonnegative, which is impossible.

Some of our reducible subgraphs

Proposition (Bi, PB)

If G has maximum average degree < 3, *then G is* ϵ*-flexibly 3-choosable.*

- A path with endpoints of degree 2 and internal vertices of degree 3
- A terminal block with maximum degree 3
- A subdivision of $K_{1,3}$ with a center of degree 4, leaves of degree 2, and internal vertices of degree 3